1. Mathematics

1.1 **Mathematics Defined!**

In Latin, and in English until around 1700, the term *mathematics* more commonly meant "astrology" (or sometimes "astronomy") rather than "mathematics"; the meaning gradually changed to its present one from about 1500 to 1800. This has resulted in several mistranslations: a particularly notorious one is Saint Augustine's warning that Christians should beware of *mathematici* meaning astrologers, which is sometimes mistranslated as a condemnation of mathematicians.

*** Mathematics** is the study of topics such as quantity (numbers), structure, space, and change. There is a range of views among mathematicians and philosophers as to the exact scope and definition of mathematics.

Mathematicians seek out patterns and use them to formulate new conjectures. Mathematicians resolve the truth or falsity of conjectures by mathematical proof. When mathematical structures are good models of real phenomena, then mathematical reasoning can provide insight or predictions about nature.

*** Through** the use of abstraction and logic, mathematics developed from counting, calculation, measurement, and the systematic study of the shapes and motions of physical objects. Practical mathematics has been a human activity for as far back as written records exist. The research required to solve mathematical problems can take years or even centuries of sustained inquiry.

Aristotle defined mathematics as "the science of quantity", and this definition prevailed until the 18th century. Starting in the 19th century, when the study of mathematics increased in rigor and began to address abstract topics such as group theory and projective geometry, which have no clear-cut relation to quantity and measurement, mathematicians and philosophers began to propose a variety of new definitions.

Some of these definitions emphasize the deductive character of much of mathematics, some emphasize its abstractness and some emphasize certain topics within mathematics. Today, no consensus on the definition of mathematics prevails, even among professionals. There is not even consensus on whether mathematics is an art or a science. A great many professional mathematicians take no interest in a definition of mathematics, or consider it indefinable. Some just say, "Mathematics is what mathematicians do."

*** Three** leading types of definition of mathematics are called logicist, intuitionist, and formalist, each reflecting a different philosophical school of thought. All have severe problems, none has widespread acceptance, and no reconciliation seems possible.

***An early** definition of mathematics in terms of logic was Benjamin Peirce's "the science that draws necessary conclusions" (1870). In the *Principia Mathematica*, Bertrand Russell and Alfred North Whitehead advanced the philosophical program known as logicism, and attempted to prove that all mathematical concepts, statements, and principles can be defined and proven entirely in terms of symbolic logic. A logicist definition of mathematics is Russell's "All Mathematics is Symbolic Logic" (1903).

Intuitionist definitions, developing from the philosophy of mathematician L.E.J. Brouwer, identify mathematics with certain mental phenomena. An example of an intuitionist definition is "Mathematics is the mental activity which consists in carrying out constructs one after the other." A peculiarity of intuitionism is that it rejects some mathematical ideas considered valid according to other definitions. In particular, while other philosophies of mathematics allow objects that can be proven to exist even though they cannot be constructed, intuitionism allows only mathematical objects that one can actually construct.

Formalist definitions identify mathematics with its symbols and the rules for operating on them. Haskell Curry defined mathematics simply as "the science of formal systems". A formal system is a set of symbols, or *tokens*, and some *rules* telling how the tokens may be combined into *formulas*. In formal systems, the word *axiom* has a special meaning, different from the ordinary meaning of "a self-evident truth". In formal systems, an axiom is a combination of tokens that is included in a given formal system without needing to be derived using the rules of the system

Rigorous arguments first appeared in Greek mathematics, most notably in Euclid's *Elements*. Since the pioneering work of Giuseppe Peano (1858–1932), David Hilbert (1862–1943), and others on axiomatic systems in the late 19th century, it has become customary to view mathematical research as establishing truth by rigorous deduction from appropriately chosen axioms and definitions.

Mathematics developed at a relatively slow pace until the Renaissance, when mathematical innovations interacting with new scientific discoveries led to a rapid increase in the rate of mathematical discovery that has continued to the present day. Mathematics is used throughout the world as an essential tool in many fields, including natural science, engineering, medicine, finance and the social sciences.

Applied mathematics, the branch of mathematics concerned with application of mathematical knowledge to other fields, inspires and makes use of new mathematical discoveries, which has led to the development of entirely new mathematical disciplines, such as statistics and game theory. Mathematicians also engage in pure mathematics or mathematics for its own sake, without having any application in mind. There is no clear line separating pure and applied mathematics, and practical applications for what began as pure mathematics are often discovered.

1.2 **Numerical systems.**

A **numeral system** (or **system of numeration**) is a writing system for expressing numbers, that is, a mathematical notation for representing numbers of a given set, using digits or other symbols in a consistent manner. It can be seen as the context that allows the symbols "11" to be interpreted as the binary symbol for *three*, the decimal symbol for *eleven*, or a symbol for other numbers in different bases.

Ideally, a numeral system will:

- Represent a useful set of numbers (e.g. all integers, or rational numbers)
- Give every number represented a unique representation (or at least a standard representation)
- Reflect the algebraic and arithmetic structure of the numbers.

For example, the usual decimal representation of whole numbers gives every non zero whole number a unique representation as a finite sequence of digits, beginning by a non-zero digit. However, when decimal representation is used for the rational or real numbers, such numbers in general have an infinite number of representations, for example 2.31 can also be written as 2.310, 2.3100000, 2.309999999..., etc., all of which have the same meaning except for some scientific and other contexts where greater precision is implied by a larger number of figures shown.

Numeral systems are sometimes called *number systems*, but that name is ambiguous, as it could refer to different systems of numbers, such as the system of real numbers, the system of complex numbers, etc.

The most commonly used system of numerals is known as Arabic numerals or Hindu–Arabic numerals. Two Indian mathematicians are credited with developing them. Aryabhata of Kusumapura developed the place-value notation in the 5th century and a century later Brahmagupta introduced the symbol for zero.

The numeral system and the zero concept, developed by the Hindus in India slowly spread to other surrounding countries due to their commercial and military activities with India. The Arabs adopted it and modified them. Even today, the

Arabs called the numerals they use "Rakam Al-Hind" or the Hindu numeral system.

The Arabs translated Hindu texts on numerology and spread it to the western world due to their trade links with them. The Western world modified them and called them the Arabic numerals, as they learnt from them. Hence the current western numeral system is the modified version of the Hindu numeral system developed in India. It also exhibits a great similarity to the Sanskrit–Devanagari notation, which is still used in India.

The simplest numeral system is the unary numeral system, in which every natural number is represented by a corresponding number of symbols. If the symbol @ is chosen, for example, then the number seven would be represented by @@@@@@@. Tally marks represent one such system still in common use.

The unary system is only useful for small numbers, although it plays an important role in theoretical computer science. Elias gamma coding, which is commonly used in data compression, expresses arbitrary-sized numbers by using unary to indicate the length of a binary numeral.

The unary notation can be abbreviated by introducing different symbols for certain new values. Very commonly, these values are powers of 10; so for instance, if * stands for one, − for ten and + for 100, then the number 304 can be compactly represented as +++ **** and the number 123 as + − − *** without any need for zero. This is called sign-value notation. The ancient Egyptian numeral system was of this type, and the Roman numeral system was a modification of this idea.

More useful still are systems which employ special abbreviations for repetitions of symbols; for example, using the first nine letters of the alphabet for these abbreviations, with A standing for "one occurrence", B "two occurrences", and so on, one could then write C+ D/ for the number 304. This system is used when writing Chinese numerals and other East Asian numerals based on Chinese. The number system of the English language is of this type ("three hundred [and] four"), as are those of other spoken languages, regardless of what written systems they have adopted.

However, many languages use mixtures of bases, and other features, for instance 79 in French is (60 + 10 + 9) and in Welsh: (4 + (5 + 10) + (3 × 20)) or (somewhat archaic) (4 × 20 − 1). In English, one could say "four score less one", as in the famous Gettysburg Address representing "87 years ago" as "four score and seven years ago".

More elegant is a *positional system*, also known as place-value notation. Again working in base-10, ten different digits 0, ..., 9 are used and the position of a digit is used to signify the power of ten that the digit is to be multiplied with, as in 304 = 3×100 + 0×10 + 4×1 or more precisely $3\times10^2 + 0\times10^1 + 4\times10^0$. Note that zero, which is not needed in the other systems, is of crucial importance here, in order to be able to "skip" a power. The Hindu–Arabic numeral system, which originated in India and is now used throughout the world, is a positional base-10 system.

Arithmetic is much easier in positional systems than in the earlier additive ones; furthermore, additive systems need a large number of different symbols for the different powers of 10; a positional system needs only ten different symbols (assuming that it uses base 10).

Positional decimal system is presently universally used in human writing. The base 1000 is also used, by grouping the digits and considering a sequence of three decimal digits as a single digit. This is the meaning of the common notation 1,000,567 used for very large numbers.

In computers, the main numeral systems are based on the positional system in base 2 (binary numeral system), with two binary digits, 0 and 1. Positional systems obtained by grouping binary digits by three (octal numeral system) or four (hexadecimal numeral system) are commonly used. For very large integers, bases 2^{32} or 2^{64} (grouping binary digits by 32 or 64, the length of the machine word) are used, as, for example, in GMP.

The numerals used when writing numbers with digits or symbols can be divided into two types that might be called the arithmetic numerals 0,1,2,3,4,5,6,7,8,9 and the geometric numerals 1, 10, 100, 1000, 10000 ..., respectively. The sign-value systems use only the geometric numerals and the positional systems use only the arithmetic numerals. A sign-value system does not need arithmetic numerals because they are made by repetition (except for the Ionic system), and a positional system does not need geometric numerals because they are made by position. However, the spoken language uses *both* arithmetic and geometric numerals.

In certain areas of computer science, a modified base-k positional system is used, called bijective numeration, with digits 1, 2, ..., k ($k \geq 1$), and zero being represented by an empty string. This establishes a bijection between the set of all such digit-strings and the set of non-negative integers, avoiding the non-uniqueness caused by leading zeros. Bijective base-k numeration is also called k-adic notation, not to be confused with p-adic numbers. Bijective base-1 is the same as unary.